Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 28 January 2013 (MN LaTeX style file v2.2)

Quadrupole moments of rotating neutron stars and strange stars

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arXiv:1301.5925v1 [astro-ph.SR] 24 Jan 2013

28 January 2013

ABSTRACT
We present results for models of neutron stars and strange stars constructed using the Hartle-Thorne slow-rotation method with a wide range of equations of state, focusing on the values obtained for the angular momentum J and the quadrupole moment Q, when the gravitational mass M and the rotational frequency Ω are specified. Building on previous work, which showed surprising uniformity in the behaviour of the moment of inertia for neutron-star models constructed with widely-different equations of state, we find similar uniformity for the quadrupole moment. These two quantities, together with the mass, are fundamental for determining the vacuum space-time outside neutron stars. We study particularly the dimensionless combination of parameters QM/J^2 (using units for which $c = G = 1$). This quantity goes to 1 in the case of a Kerr-metric black hole and deviations away from 1 then characterize the difference between neutron-star and black-hole space-times. It is found that QM/J^2 for both neutron stars and strange stars decreases with increasing mass, for a given equation of state, reaching a value of around 2 (or even less) for maximum-mass models, meaning that their external space-time is then rather well approximated by the Kerr metric. If QM/J^2 is plotted against compactness $R/2M$ (where R is the radius), it is found that the relationship is nearly unique for neutron-star models, independent of the equation of state, while it is significantly different for strange stars. This gives a new way of possibly distinguishing between them.

Key words: stars: neutron – stars: rotation.

1 INTRODUCTION

Neutron stars are compact objects consisting of closely-packed neutrons together with protons and electrons (and at high densities also muons and possibly other particles such as hyperons, pions and kaons). Their central densities reach values where the microphysics is not well understood and so they could serve as laboratories for investigating the behaviour of nuclear matter under extreme conditions. A wide range of approaches has been used for modeling neutron stars and comparison between derived properties of the models and observations has enabled constraints to be placed on prescriptions for the nucleon–nucleon interactions (Haensel et al. 2007; Lattimer & Prakash 2007; Akmal et al. 1998; Rikovska Stone et al. 2003; Urbanec et al. 2010; Gandolfi et al. 2010; Newton & Stone 2009). An important, and widely discussed, alternative to the standard neutron-star picture is given by the possibility that some or all of the matter may consist of deconfined quarks. The most radical version of this, the strange star picture, envisages a star consisting entirely of deconfined quarks (apart, perhaps, from a thin crust of normal matter) and has its origin in the suggestion by Witten (1984) that such matter consisting of nearly equal numbers of up, down and strange quarks might represent the absolute ground state for strongly-interacting matter, even down to zero pressure. Strange-star models have been investigated and discussed by many authors (Haensel et al. (1986); Alcock et al. (1986); Alford et al. (1999); Farhi & Jaffe (1984); Colpi & Miller (1992) etc.). If these strange stars exist, they would have unusual properties because of being bound together by a combination of the strong force and gravity, rather than

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just by gravity as is usually the case. It is then important to look for features of the internal and external space-time structure of compact stars, which might enable one to distinguish observationally between standard neutron stars and possible strange stars.

Much of what we know about the rotation speeds of compact stars comes from observations of radio pulsars, thought to be rotating magnetized neutron stars emitting dipole radiation. Observed frequencies range from ~ 0.1Hz up to $\sim$kHz (Manchester et al. 2005; Ghosh 2007). The fastest currently-known pulsar is PSR J1748-2446ad (Hessels et al. 2006) rotating with frequency of 716 Hz. Studying the rotational properties of compact stars is of great interest and several different approaches have been used for this within the context of general relativity. The first one involved the perturbative slow rotation approximation (taken to 2nd order in the neutron star angular velocity Ω) as investigated by Hartle (1967); Hartle & Thorne (1968); Chandrasekhar & Miller (1974) and many others. The slow rotation approximation gives equations governing the structure of the rotating objects allowing one to calculate general properties such as the mass M, angular momentum J and quadrupole moment Q. These three parameters represent all that is necessary in order to characterize the external gravitational field within the slow-rotation approximation if one is retaining only terms in the perturbative expansion around the underlying non-rotating comparison object up to second order in the angular velocity Ω. Subsequently various finite-difference schemes have been developed for calculating models of rapidly rotating compact stars (Butterworth & Ipser (1975); Komatsu et al. (1989); Cook et al. (1994); Salgado et al. (1994); Stergioulas & Friedman (1995); Nozawa et al. (1998), etc.) and there is a freely downloadable code RNS (Stergioulas 1997) that can be used for calculating rapidly rotating neutron star models. Spectral methods have also been used and codes implementing these are available as a part of the `LORENE` package (Bonazzola et al. 1998). These have been used in various contexts (Lo & Lin (2011); Haensel et al. (2009); Bejger et al. (2010); Gondek-Rosińska et al. (2001); Amsterdamski et al. (2002), etc.) and have been compared with other numerical schemes by Stergioulas (2003); Berti et al. (2005); Nozawa et al. (1998).

Properties of rotating compact stars were investigated in pioneering works by Hartle & Thorne (1968); Chandrasekhar & Miller (1974); Miller (1977) and others. Quadrupole moments were discussed by Miller (1977); Laarakkers & Poisson (1999) and others; recently Bradley & Fodor (2009) discussed the difference between the quadrupole moments of neutron stars and Kerr black holes. Some differences between rotating neutron stars and strange stars were pointed out by Bagchi (2010). The impact of the properties of neutron stars and strange stars on the behaviour of the external space-time and its astrophysically-important properties has been discussed by Torok et al. (2008); Haensel et al. (2009); Bejger et al. (2010); Gondek-Rosińska et al. (2001); Amsterdamski et al. (2002), etc.

In previous works, several authors (Lattimer & Prakash 2001; Bejger & Haensel 2002) have shown that the factor I/MR^2, where $I = J/\Omega$ is the moment of inertia, follows an almost unique relation in terms of $R/2M$ for most neutron-star equations of state whereas an entirely different behaviour was found for strange stars with an equation of state based on the MIT bag model. Motivated by these results, we present here a corresponding discussion for the quadrupole moments, finding again similar behaviour for the quantity QM/J^2. Throughout the paper, we use units for which $c = G = 1$.

2 MODELS OF COMPACT STARS

2.1 Equations of state

We focus here on two families of compact objects. Standard neutron star models are taken to be composed of neutrons in β-equilibrium with protons and electrons (and eventually muons and hyperons at higher densities) while strange star models, are taken to be composed of u,d and s quarks, (see, e.g. Haensel et al. (2007) for general overview of standard neutron-star models and Witten (1984); Haensel et al. (1986); Alcock et al. (1986) for details of the strange star hypothesis and models).

We use here a wide range of equations of state for standard neutron-star matter, based on various different assumptions and methodologies. Variational theory is represented by the widely-used equation of state of Akmal et al. (1998): we use the model originally labelled as A18 + δv + UIX* with the Argone 18 potential, including three-body forces and relativistic boost corrections. Relativistic mean field theory used to fit results obtained with direct Dirac-Brueckner-Hartree-Fock calculations is represented by the UBS equation of state (Urbanec et al. 2010). As representative of Brueckner-Hartree-Fock theory the equation of state of Baldo et al. (1997) labelled BBB2 is chosen. As a representative parametrization of the Skyrme potentials, we take the SLy4 equation of state - see e.g. Rikovska Stone et al. (2003) for detailed study of the Skyrme potential. As representative of the energy density functional developed by Bombaci (1995), model BPAL12 is used. We also include the relatively new equation of state of Gandolfi et al. (2010) based on the auxiliary field diffusion Monte Carlo (AFDMC) technique, which we label as Gandolfi. Equations of state including hyperons are represented by by those labelled as BALBN1H1 (Balberg & Gal 1997) and GLENDH3 (Glendenning 1985)[1].

[1] For models BALBN1H1, BBB2, BPAL12, BALBN1H1, GLENDNH1 we used equation of state tables from the public domain code LORENE (Bonazzola et al. 1998). The APR equation of state was calculated using the effective Hamiltonian given in the original paper

As an equation of state for strange stars we are using the standard form of the basic MIT Bag model (Chodos et al. 1974) where the pressure P and baryon number density $n_{\rm B}$ are related to the energy density $\mathcal{E}$ via the equations

$$P = \frac{1}{3}\left(\mathcal{E} - 4B\right),$$
$$n_{\rm B} = \left[\frac{4(1-2\alpha_{\rm c}/\pi)^{1/3}}{9\pi^{2/3}\hbar}(\mathcal{E}-B)\right]^{3/4},$$

where B is the bag constant, whose value is related to the energy density of matter at zero pressure by $\mathcal{E} = 4B$, and $\alpha_{\rm c}$ is the strong interaction coupling constant. We here use the MIT Bag Model with standard parameters $B = 10^{14}$g.cm^{-3} and $\alpha_{\rm c} = 0.15$, and we also use twice this value for B in order to see its impact on the properties of the calculated models. The Bag Model gives a very simple representation of the matter in strange stars, but more sophisticated descriptions generally give very similar results for the main quantities of interest for this paper.

We have chosen here equations of state representing different approaches to the microphysics of neutron star matter, but it is important to note that differences between different versions (or parametrizations) of the same approach can also be very significant, (see e.g. Rikovska Stone et al. (2003) for discussion of different parametrizations of Skyrme potential). We should stress here that our aim in selecting this set of equations of state was to choose a broad representative sample coming from a range of different microphysical approaches, without yet considering closer constraints coming from recent observations, such as those coming from the double pulsar (Podsiadlowski et al. 2005), from the new highest-mass neutron stars (Demorest et al. 2010; Antoniadis et al 2013) and from X-ray bursts (Özel et al. 2010; Steiner et al. 2010). Our aim here is to show that even with taking this very broad set of equations of state, we see a surprising degree of convergence in some of the results obtained.

2.2 The Hartle–Thorne method

Hartle and Thorne (Hartle 1967; Hartle & Thorne 1968) developed a method for calculating models of rotating neutron stars using a slow rotation approximation that is valid for suitably small angular velocities with $\Omega^2 \ll GM/R^3$. It has been shown that this approximation can be used with good accuracy for almost all currently observed neutron stars, even for most millisecond pulsars (Stergioulas 2003; Berti et al. 2005). While it is, of course, of great interest to calculate very rapidly rotating models, the Hartle–Thorne method, when correctly used within its strict range of validity, is the most accurate method available, and it is appropriate for the vast majority of known neutron stars.

The space-time metric around and inside rotating compact objects is given in the slow rotation approximation by the perturbation of a spherically symmetric metric retaining terms up to second order in the angular velocity Ω (as measured from infinity); this can be expressed in the form

$$\begin{aligned} {\rm d}s^2 &= -{\rm e}^{2\nu}[1+2h_0(r)+2h_2(r)P_2]{\rm d}t^2 + {\rm e}^{2\lambda}\left\{1+\frac{{\rm e}^{2\lambda}}{r}[2m_0(r)+2m_2(r)P_2]\right\}{\rm d}r^2 \\ &+ r^2\left[1+2k_2(r)P_2\right]\left\{{\rm d}\theta^2 + [{\rm d}\phi - \omega(r){\rm d}t]^2\sin^2\theta\right\}. \end{aligned} \tag{1}$$

One can see that the perturbation away from the spherically symmetric non-rotating solution involves all of the metric functions $g_{tt},\ g_{rr},\ g_{\theta\theta}$ and $g_{\phi\phi}$[2] and that an additional term $g_{t\phi}$ appears, describing the dragging of inertial frames. The perturbation functions $h_0(r), h_2(r), m_0(r), m_2(r), k_2(r)$ are quantities of order Ω^2 and are functions only of the radial coordinate r , while $\omega(r)$ is of order Ω. The deviation away from spherical symmetry in the diagonal part of metric is given by the 2nd order Legendre polynomial

$$P_2 = P_2(\cos\theta) = (3\cos^2\theta - 1)/2. \tag{2}$$

All of the perturbation functions must be calculated with appropriate boundary conditions at the centre and at the surface of the configuration. The second-order ones are labelled with a subscript indicating the multipole order of the perturbative quantities: $l = 0$ for the monopole (spherical) deformations and $l = 2$ for quadrupole deformations representing the deviation away from spherical symmetry. Equations for the perturbation functions are derived from the Einstein field equations $G^{\mu\nu} = 8\pi T^{\mu\nu}$. If we put zero on the right hand side of these, we get the equations for the external vacuum space-time and then we can express the perturbation functions in terms of properties of the central object as measured by a distant observer: the mass M, angular momentum J and quadrupole moment Q (Hartle & Thorne 1968; Abramowicz et al. 2003; Chandrasekhar & Miller 1974). We take the energy-momentum tensor to be that of a perfect fluid

$$T^{\mu\nu} = (\mathcal{E}+P)U^\mu U^\nu + Pg^{\mu\nu}, \tag{3}$$

(Akmal et al. 1998) and the UBS model represents the parametrization H in the original paper (Urbanec et al. 2010). The table for SLy4 was kindly provided by Jiřina Říkovská Stone. The Gandolfi equation of state was kindly provided by authors of original paper.

[2] The subscript 0 in the metric functions ν and λ refers to the unperturbed Schwarzschild geometry.

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where $\mathcal{E}$ is the energy density, P is the pressure and U^μ is the fluid four-velocity. We deal with models rotating uniformly with angular velocity Ω, and the non-zero components of the four-velocity are then

$$U^t = \left[-(g_{tt} - 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})\right]^{1/2}, \quad U^\phi = \Omega U^t. \tag{4}$$

The method used for calculating neutron-star models and obtaining the integral properties (M, J and Q), starting from specifying the central density and the rotation frequency, has been discussed by Hartle (1967); Hartle & Thorne (1968); Chandrasekhar & Miller (1974); Miller (1977). We use the same procedure here.

We start by solving the equations for the unperturbed non-rotating objects. These are the equations of hydrostatic equilibrium, and the equations for the metric functions

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -(\mathcal{E}+P)\frac{m(r)+4\pi r^3 P}{r(r-2m(r))}, \tag{5}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\mathcal{E}r^2, \tag{6}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{m(r)+4\pi r^3 P}{r(r-2m(r))}, \tag{7}$$

$$\mathrm{e}^{2\lambda} = \left(1-\frac{2m(r)}{r}\right)^{-1}, \tag{8}$$

with appropriate boundary conditions at the centre of the star. For starting the integration, we need to have expressions for the values of the dependent variables a little away from the centre. These are obtained by making series expansions, leading to

$$P \rightarrow P_\mathrm{c} - (2\pi/3)(P_\mathrm{c}+\mathcal{E}_\mathrm{c})(3P_\mathrm{c}+\mathcal{E}_\mathrm{c})r^2, \tag{9}$$

$$m \rightarrow 4/3\pi\mathcal{E}_\mathrm{c}r^3, \tag{10}$$

$$\nu \rightarrow \nu_\mathrm{c} + (2\pi/3)(P_\mathrm{c}+\mathcal{E}_\mathrm{c})r^2, \tag{11}$$

$$\mathrm{e}^{2\lambda} \rightarrow \left[1-(8/3)\pi\mathcal{E}_\mathrm{c}r^2\right]^{-1}. \tag{12}$$

The subscript c denotes the value of the quantity at the centre; ν_c is initially unknown, but can be calculated by changing the variable to $\nu-\nu_\mathrm{c}$ and then matching the interior and exterior solutions at the surface of the star.

For considering the rotational perturbations, we follow the earlier work in comparing rotating and non-rotating models having the same central density. Inserting the form of the Hartle–Thorne metric (1) and the energy momentum tensor for a perfect fluid (3) into the Einstein field equations then leads to differential equations for perturbation functions. We start with the one coming from the $(t\phi)$ component

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right) + 4\frac{\mathrm{d}j}{\mathrm{d}r}\tilde{\omega} = 0, \tag{13}$$

where $j = \mathrm{e}^{-(\lambda+\nu)}$ and $\tilde{\omega} = \Omega - \omega$. Near to $r=0$, $\tilde{\omega}$ has following behaviour:

$$\tilde{\omega} \rightarrow \tilde{\omega}_\mathrm{c} + \frac{8\pi}{5}\left(\mathcal{E}_\mathrm{c}+P_\mathrm{c}\right)\tilde{\omega}_\mathrm{c}r^2, \tag{14}$$

where $\tilde{\omega}_\mathrm{c}$ is a constant whose value can again be found by matching with the exterior solution. Outside the star, one has

$$\tilde{\omega}(r) = \Omega - \frac{2J}{r^3}, \tag{15}$$

where the constant J is the angular momentum of the rotating neutron star (Hartle 1967). The angular momentum is then given by the formula

$$J = \frac{R^4}{6}\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)_{r=R}, \tag{16}$$

and both $\tilde{\omega}_\mathrm{c}$ and J can then be calculated by matching $\tilde{\omega}$ and $\mathrm{d}\tilde{\omega}/\mathrm{d}r$ at the surface with their exterior solutions, if Ω is specified. The moment of inertia I is given by the standard relation $I = J/\Omega$.

The other field equations lead to the following differential equations for the $l=0$ and $l=2$ perturbation functions. The $l=0$ functions are given by

$$\frac{\mathrm{d}m_0}{\mathrm{d}r} = 4\pi r^2(\mathcal{E}+P)\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}P}p_0 + \frac{1}{12}r^4 j^2\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 - \frac{1}{3}r^3\tilde{\omega}^2\frac{\mathrm{d}j^2}{\mathrm{d}r}, \tag{17}$$

$$\begin{aligned}\frac{\mathrm{d}p_0}{\mathrm{d}r} &= -\frac{\mathrm{d}h_0}{\mathrm{d}r} + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3 j^2\tilde{\omega}^2}{r-2m(r)}\right)\\ &= -\frac{m_0(1+8\pi r^2 P)}{(r-2m(r))^2} - \frac{4\pi(\mathcal{E}+P)r^2}{r-2m(r)}p_0 + \frac{1}{12}\frac{r^4 j^2}{r-2m(r)}\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3 j^2\tilde{\omega}^2}{r-2m(r)}\right).\end{aligned} \tag{18}$$

Near to $r=0$, m_0 and p_0 have the following behaviour:

$$
\begin{aligned}
m_0 &\rightarrow \frac{4\pi}{15}(\mathcal{E}_c+P_c)\left[\left(\frac{d\mathcal{E}}{dP}\right)_c+2\right]j_c^2\tilde{\omega}_c^2 r^5, &(19)\\
p_0 &\rightarrow \frac{1}{3}j_c^2\tilde{\omega}_c^2 r^2, &(20)
\end{aligned}
$$

while outside the star, where $\mathcal{E}=P=0$, $m(r)=m(R)=M_0$ and $j=1$, they are given by

$$
\begin{aligned}
m_0 &= \delta M - \frac{J^2}{r^3}, &(21)\\
h_0 &= -\frac{\delta M}{r-2M_0} + \frac{J^2}{r^3(r-2M_0)}, &(22)
\end{aligned}
$$

where δM is a constant giving the change in gravitational mass resulting from the rotation. For the $l=2$ perturbation functions, where we use $v_2 = h_2 + k_2$ instead of k_2, we have

$$
\begin{aligned}
\frac{dv_2}{dr} &= -2\frac{d\nu_0}{dr}h_2 + \left(\frac{1}{r}+\frac{d\nu_0}{dr}\right)\left[\frac{1}{6}r^4 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2 - \frac{1}{3}r^3\tilde{\omega}^2\frac{dj^2}{dr}\right], &(23)\\
\frac{dh_2}{dr} &= -\frac{2v_2}{r(r-2m(r))d\nu_0/dr} + \left\{-2\frac{d\nu_0}{dr} + \frac{r}{2(r-2m(r))d\nu_0/dr}\left[8\pi(\mathcal{E}+P) - \frac{4m(r)}{r}\right]\right\}h_2\\
&+ \frac{1}{6}\left[r\frac{d\nu_0}{dr} - \frac{1}{2(r-2m(r))d\nu_0/dr}\right]r^3 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2 - \frac{1}{3}\left[r\frac{d\nu_0}{dr} + \frac{1}{2(r-2m(r))d\nu_0/dr}\right]r^2\tilde{\omega}^2\frac{dj^2}{dr}. &(24)
\end{aligned}
$$

and $m_2(r)$ is given by

$$
\frac{m_2}{r-2m(r)} = -h_2 + \frac{1}{6}r^4 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2 - \frac{1}{3}r^3\tilde{\omega}^2\frac{dj^2}{dr}. \qquad (25)
$$

The solutions of equations (23,24) are expressed as the sum of a particular integral and a complementary function

$$
\begin{aligned}
h_2 &= h_2^{(P)} + A h_2^{(C)}, &(26)\\
v_2 &= v_2^{(P)} + A v_2^{(C)}, &(27)
\end{aligned}
$$

where A is a constant to be determined and the homogeneous equations for complementary functions are

$$
\begin{aligned}
\frac{dv_2^{(C)}}{dr} &= -2\frac{d\nu_0}{dr}h_2^{(C)}, &(28)\\
\frac{dh_2^{(C)}}{dr} &= -\frac{2v_2^{(C)}}{r(r-2m(r))d\nu_0/dr} + \left\{-2\frac{d\nu_0}{dr} + \frac{r}{2(r-2m(r))d\nu_0/dr}\left[8\pi(\mathcal{E}+P) - \frac{4m(r)}{r}\right]\right\}h_2^{(C)}. &(29)
\end{aligned}
$$

Near to $r=0$ particular integrals have the behaviours

$$
\begin{aligned}
h_2^{(P)} &\rightarrow ar^2, &(30)\\
v_2^{(P)} &\rightarrow br^4, &(31)
\end{aligned}
$$

where a and b are constants which are related by

$$
b + \frac{2\pi}{3}(\mathcal{E}_c + 3P_c)a = \frac{2\pi}{3}(\mathcal{E}_c + P_c)j_c^2, \qquad (32)
$$

while the complementary functions have the behaviours

$$
\begin{aligned}
h_2^{(C)} &\rightarrow Br^2, &(33)\\
v_2^{(C)} &\rightarrow -\frac{2\pi}{3}(\mathcal{E}_c + 3P_c)Br^4, &(34)
\end{aligned}
$$

where B is another constant. The integrations are carried out with arbitrarily assigned values of a and B.

Outside the star, h_2 and v_2 take the form

$$
\begin{aligned}
h_2 &= J^2\left(\frac{1}{M_0 r^3} + \frac{1}{r^4}\right) + K\mathcal{Q}_2^2\left(\frac{r}{M_0}-1\right), &(35)\\
v_2 &= \frac{J^2}{r^4} + K\frac{2M_0}{[r(r-2M_0)]^{1/2}} + \mathcal{Q}_2^1\left(\frac{r}{M_0}-1\right), &(36)
\end{aligned}
$$

where K is a constant and the $\mathcal{Q}_a^b$ are associated Legendre functions of the second kind (see equations (137) and (141) of the original paper Hartle (1967) for explicit formulae). The constant K can then be calculated by matching the internal values for v_2 and h_2 to the external ones at the surface.

Once these integrations have been carried out, the total mass of the rotating star is given as

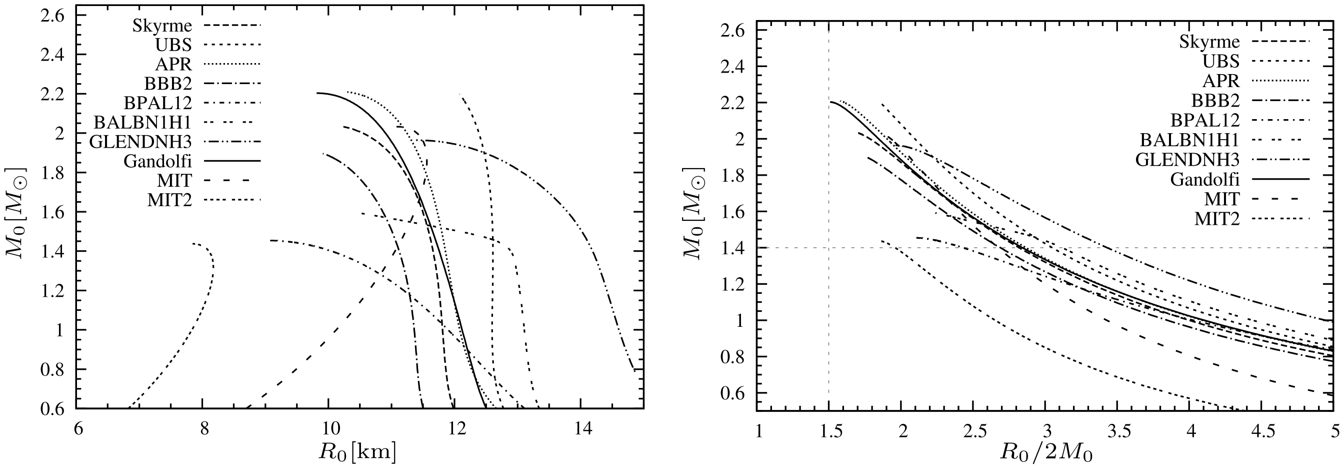


Figure 1. Mass versus radius *(Left)* and compactness $x = R_0/2M_0$ *(Right)* for non-rotating models with the selected equations of state.

$$M = M_0 + \delta M = M_0 + m_0(R) + J^2/R^3, \tag{37}$$

and its quadrupole moment is

$$Q = \frac{J^2}{M} + \frac{8}{5}KM^3. \tag{38}$$

3 RESULTS

3.1 Static non-rotating configurations

We here denote integral quantities for the non-rotating object using the subscript 0 (i.e. M_0 is the mass of the non-rotating star and R_0 is its radius) and we define the compactness parameter x of the star as its actual (circumferential) radius divided by its Schwarzschild radius

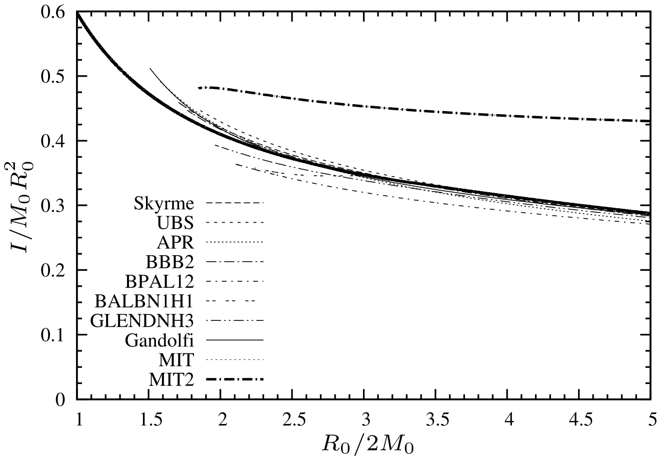
$$x = \frac{R_0}{2M_0} \tag{39}$$

The calculated mass–radius relations for the selected equations of state, for both neutron stars and strange stars, are plotted in Fig. 1 *(Left)* while Fig. 1*(Right)* shows the resulting dependencies of mass on the compactness x. The vertical line in Fig. 1*(Right)* corresponds to the condition $R_0 = 3M_0$ which is boundary of the situations where the neutron star has or does not have unstable circular null geodesics external to its surface. We can see that none of the selected equations of state allows the existence of "extremely compact objects" with $R_0 < 3M_0$ which would contain trapped null geodesics in their interior and thus could cool via the non-standard scenario suggested by Stuchlík et al. (2009); however, the Gandolfi et al. (2010) equation of state allows configurations very close to that condition. The horizontal line represents the canonical neutron star mass $M = 1.4\,M_\odot$. We can see that, excluding MIT2, this mass corresponds here to the interval ≈ 2.4 to ≈ 3.5 in compactness and so to a range of radii R_0 of (4.8–7.0) M_0. Note also that according to Fig. 1*(Right)*, assuming the most astrophysically interesting masses of neutron stars, (M greater than around $1.25M_\odot$), the corresponding compactness is $x < 4$ i.e. $R_0 < 8M_0$.

Concerning Fig. 1*(Left)*, we stress again that the range of equations of state taken here is extremely wide, and that some of these mass/radius curves are not consistent with the observational constraints mentioned earlier, although it should be pointed out that those constraints apply to certain particular objects and the equations of state which they seem to exclude could still be valid for other ones. It is by no means certain that one single equation of state (or microphysical model) would apply for all of these objects.

3.2 Moments of inertia

The moment of inertia of a neutron star model is calculated from the angular momentum using the standard formula $I = J/\Omega$. Since the angular momentum J, as calculated here, is of first order in the angular velocity Ω (with the rotational shape correction at order Ω^3 being neglected), dividing it by Ω leads to a quantity which does not vary with Ω and depends only on the structure of the spherical non-rotating comparison object on which the perturbative expansion is based. It was shown by Lattimer & Prakash (2001); Bejger & Haensel (2002) that, for standard neutron stars, the dimensionless factor $I/M_0R_0^2$



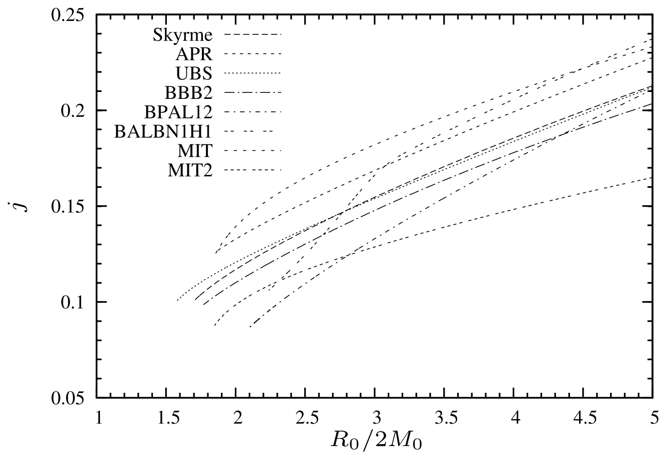


Figure 2. *Left:* The moment of inertia factor $I/M_0R_0^2$ plotted versus compactness for the selected equations of state. The curve given by the "universal" formula is shown with the heavy solid line. *Right:* The angular momentum parameter $j = J/M_0^2$ for objects rotating at 300 Hz plotted versus compactness $x = R_0/2M_0$.

could be related to the compactness by a "universal" formula which is almost independent of the equation of state. In terms of our compactness parameter $x = R_0/2M_0$,

$$I/M_0R_0^2 = \frac{1}{0.295x + 2}, \quad x \geqslant 3.33,$$
$$I/M_0R_0^2 = \frac{2x + 3.38}{9x}, \quad x < 3.33. \tag{40}$$

This relation is plotted with the heavy solid line in the left frame of Fig. 2 and it can be seen that it fits rather well with the curves for the standard neutron star equations of state, despite the diversity of their formulations, whereas the curves for the strange stars are quite different (the two strange-star curves lie almost exactly on top of each other, despite having different values for the bag constant). The strange star moment of inertia factor $I/M_0R_0^2$ for large values of x (i.e. for low mass strange stars), tends towards the value 2/5 which is the well-known result in classical physics for uniform density spheres. This is not surprising since these models do indeed have almost constant density profiles with the central energy density being close to the surface value, which also results in them having masses roughly following $M_0 \propto R_0^3$.

The right-hand frame of Fig. 2 shows the angular momentum parameter $j = J/M_0^2$ (J/M^2 taken to lowest order) plotted as a function of compactness for objects rotating at 300 Hz, which is around the maximum frequency for which the slow-rotation approximation can be considered as reliable for a neutron star with canonical mass and radius. (Note that this usage of j is not to be confused with that in section 2.2; we are caught here between two conflicting notations.) Since this j depends linearly on the rotational frequency, it is simple to calculate its value for any other rotational frequency, e.g. one tenth of the frequency would lead to one tenth of the value of j for a given equation of state and compactness[3].

3.3 Quadrupole moments

The leading order contribution to the quadrupole moment (and the only one retained within our slow-rotation approximation) is of 2nd order in the angular frequency Ω. A suitable dimensionless and frequency-independent quantity characterizing this is provided by the Kerr factor $\tilde{q} = QM_0/J^2$ (QM/J^2 taken to lowest order) (Miller 1977) that can be used to represent the deviations of the external Hartle–Thorne metric away from the Kerr black hole metric (for which $\tilde{q} = 1$). For a given equation of state, the value of $\tilde{q}$ is fully determined by the central density or pressure of the star or by its corresponding compactness x. Computed values of $\tilde{q}$ for different degrees of compactness are shown in Fig. 3 for both neutron stars and strange stars and it can be seen that, once again, there is an almost universal relationship between $\tilde{q}$ and compactness for the neutron star models, while the relation for the strange stars is quite different.

We have again looked for a suitable analytic fitting formula to represent this almost universal behaviour for the neutron-star models. For doing this, we selected a linear dependency for the higher values of x and a quadratic one for the lower values, with the minimum corresponding to $x = 1$, $\tilde{q} = 1$, i.e. we use the relations

$$\tilde{q} = a_1 x + a_0, \quad x > x_0,$$

[3] This statement is of course true if the compactness is replaced with any parameter of the non-rotating neutron star, e.g. with its mass M_0.

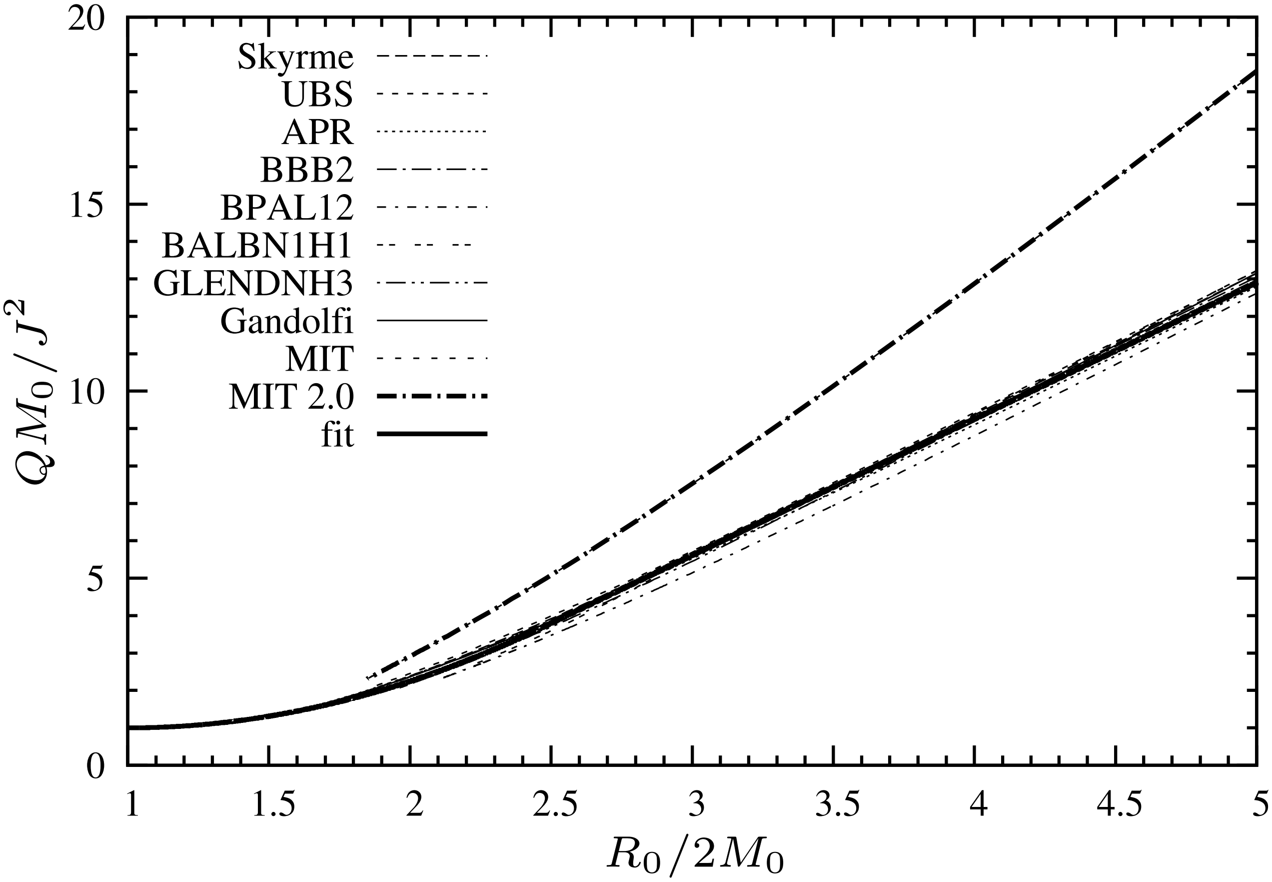


Figure 3. The Kerr factor $\tilde{q} = QM_0/J^2$ plotted against compactness for the selected equations of state. The approximate analytic relation is labelled as "fit" and is shown using the bold solid line.

$$\tilde{q} \quad = \quad b(x-1)^2 + 1, \quad x \leqslant x_0, \tag{41}$$

where a_1 and a_0 are fitted parameters while b and x_0 are calculated assuming that the function is continuous and smooth at the point x_0 where the functions are matched. Under these circumstances, the relation for b is

$$b = {a_1}^2/[4.(-a_1 - a_0 + 1)], \tag{42}$$

and the matching point x_0 is given by

$$x_0 = \frac{2(1-a_0)}{a_1} - 1. \tag{43}$$

We have found that for $a_1 = 3.64$ and $a_0 = -5.3$, the analytic relation fits the calculated values very well, as shown in Fig. 3.

It can be seen from Fig. 3 that $\tilde{q}$ is systematically decreasing for increasingly compact models and seems to be tending towards the Kerr value $\tilde{q} = 1$ as the non-rotating comparison star gets closer to becoming a Schwarzschild black hole, i.e. as $x \to 1$ or $R_0 \to 2M_0$. This also corresponds to the stellar models getting closer to the maximum mass limit. The upper limit for $\tilde{q}$, for the most astrophysically interesting neutron-star models, is about 9. For the strange stars of given compactness, $\tilde{q}$ is always larger than it is for neutron stars of the same compactness, but the tendency towards $\tilde{q} = 1$ as $R_0 \to 2M_0$ is seen for both families of objects. We find that $\tilde{q}$ is almost identical for both values of the bag constant, similar to the situation for the moment of inertia factor.

A key feature of the results is the systematic decrease of $\tilde{q}$ as the mass increases and approaches its maximal value for any given equation of state, as shown in Fig. 4. We also note that for compact stars with the canonical mass $M = 1.4M_\odot$, the value of $\tilde{q}$ depends strongly on the assumed equation of state (as a result of the varying compactness of the models) and its value can range between $\tilde{q} \simeq 2$ and $\tilde{q} \simeq 7-8$.

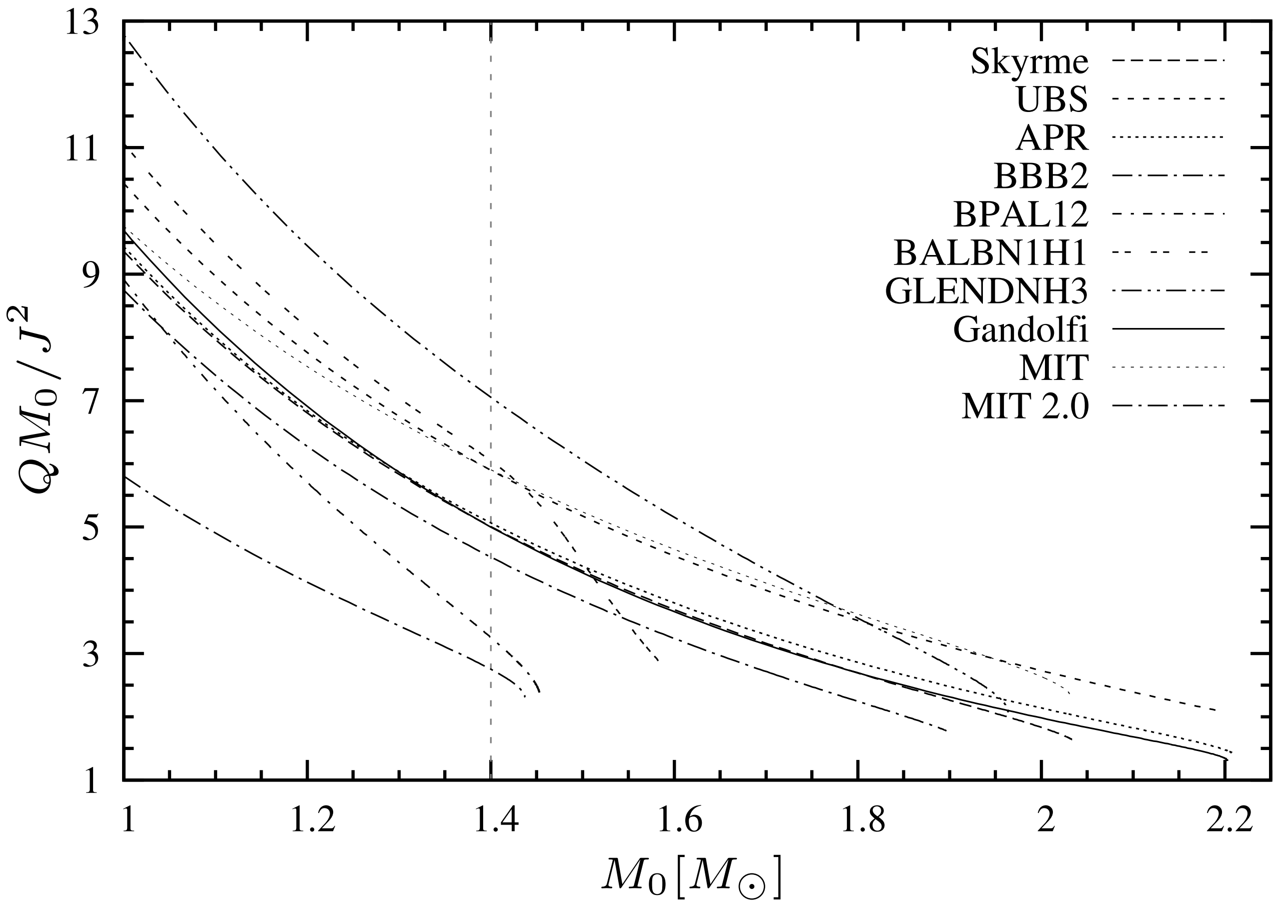


Figure 4. The Kerr factor $\tilde{q} = QM_0/J^2$ is plotted against the mass of neutron stars (and strange stars) for the selected equations of state.

4 CONCLUSIONS

In this paper, we have calculated models of rotating neutron stars and strange stars within the framework of the Hartle–Thorne slow-rotation approximation for a variety of equations of state, in order to investigate the behaviour of the parameters determining the external space-time. We have shown that for the most astrophysically interesting cases of objects with masses greater than $\sim 1.25M_\odot$, the compactness parameter $x = R_0/2M_0$ is less than ~ 4 in the case of neutron stars and less than ~ 3 for strange stars. For these models the angular momentum parameter $j = J/M_0^2$ corresponding to the maximum currently-observed rotation speed of 716 Hz would come in the interval 0.2–0.5, if calculated with the slow-rotation approximation, depending on the mass and equation of state. For the lower part of this range, the slow-rotation approximation could be used consistently even for this rather rapidly rotating object. Most observed neutron stars can be accurately treated within the more-or-less analytic slow-rotation approximation without needing a more elaborate numerical treatment.

Even if different equations of state for standard neutron-star matter give rather different neutron-star properties, some combinations of neutron-star parameters can be very accurately approximated using just the compactness of the neutron star. That this is so for the moment of inertia factor I/MR^2 was previously known; we have here demonstrated that the same is true for the Kerr factor $\tilde{q} = QM/J^2$ for which we have presented a new analytic fit. The interval for possible values of $\tilde{q}$ for the most astrophysically interesting neutron-star models ranges from $\tilde{q} \sim 1.5$ for the most extreme objects close to maximal mass up to $\tilde{q} \sim 9$ for low mass objects. The analytical representations of the key parameters enable us to express the space-time metric around a slowly-rotating neutron star in terms of its mass M, radius R and rotational frequency f.

The "universal" relation between the Kerr factor $\tilde{q}$ and the compactness x of the neutron stars is almost independent of the equation of state. However, the equivalent relation between $\tilde{q}$ and x for the strange stars is significantly different from this. If it becomes possible to constrain both $\tilde{q}$ and x independently from observations, this could be used as a way of indicating whether strange stars may actually exist. We believe that this is another significant new result, to set alongside realising that for compact stars (both neutron stars and strange stars) with masses close to the maximum allowed for the given equation of state, the external space-time is very close to the standard Kerr space-time, a fact that can considerably simplify the modelling of accretion and optical phenomena in the vicinity of these compact stars.

The quadrupole moment can also play a key role in determining the position of the ISCO, especially for low-mass compact stars. This can have an impact on the validity of QPO models as discussed by Urbanec et al. (2010) and so it is necessary to include the quadrupole moment in the analysis of observational data, especially for masses that are close to the observed values (Török et al. 2010, 2012). In a forthcoming paper, we plan to study the astrophysical consequences of our results and to compare them with those obtained by other methods.

ACKNOWLEDGMENTS

This work has been supported by the Czech grants MSM 4781305903, LC 06014 and GAČR P209/12/P740. The authors further acknowledge an internal student grant from the Silesian University in Opava (SGS/1/2012) and the project CZ.1.07/2.3.00/20.0071 "Synergy" in the frame of Education for Competitiveness Czech Operational Programme supporting the international collaborations of the Opava Institute of Physics. The work has also been supported by the CompStar research networking programme of the European Science Foundation.

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